PRODUCT BLOCK FOR VOLterra INTEGRAL EQUATIONS OF THE
SECOND KIND WITH SINGULAR KERNEL

by
G. M. Attia(1), A. E. Huseen(2), S. H. El-Emam(3), and S. F. El-Kotb(4)

Abstract: The Block method is a good self-starting method for solving Volterra Integral Equations
with continuous kernel. In this paper two modifications are suggested to develop Block method and to use
it to solve Volterra Integral Equations of the second kind with singular kernel. The idea of the
product technique is used for Block method, which is the first modification, while the second
modification is the use of the graded nodes with Block method. Some algorithms are considered and
the solution procedures are carried out. Implementation and testing of the considered algorithms have
been done. The results show that the graded nodes give good results compared with equal spaced
nodes.

Keywords: Volterra Integral Equation, Product Integration, Singular Kernel, Block Method.

1. Introduction: Mathematical modeling processes in the biological and the
physical applications (see Brunner [4]) lead quite frequently to Volterra
Integral Equation of the second kind of the following form:
\[ y(x) = f(x) + \int_0^x \frac{p(\alpha, \beta)}{\alpha} y(t, x) \, dt, \quad x \in [0,1] \]

where \( p(\alpha, \beta) \) refers to the type of singularity used with initial condition of
\( y(0) = f(0) \) and \( \frac{1}{\alpha} \) is smooth on the domain \( 0 \leq t \leq x \leq 1 \). In this, paper two
types of singularities are considered:
1- \( p(x, t) = (x-t)^{\alpha} \) which has a singularity at \( x = t \), \( 0 < \alpha < 1 \)
2- \( p(x, t) = (x-t)^{\alpha} \) \( \frac{1}{\alpha} \), \( 0 < \alpha < 1 \) which has two types of singularities, the first at
\( t = 0 \) and the other is moving along the line \( x = t \). The commonly encountered
values of \( \alpha \) are \( (0 \text{ and } 1/2) \), but other (rational) values (e.g., \( \alpha = 2/3 \)) are also
known to occur, for example in the modeling of the flow of a hot gas through a
metallic tube, with reaction, arising at the walls of the tube; (see Delves and
Walsh [8], Atkinson [1] and Clàus [5]). In this paper the product Block is
considered.

(1) Lecturer of Eng. Mathematics, Faculty of Engineering, Mansoura University.
(2) Prof. of Eng. Mathematics, Faculty of Engineering, Cairo University.
(3) Prof. of Mech. Power Engineering, Faculty of Engineering, Mansoura University.
(4) M. Sc. in Eng. Mathematics, Faculty of Engineering, Mansoura University.

Accepted May 20, 1998
2. Mathematical Preliminaries: The singular Volterra integral equation of the second kind:

\[ y(x) = f(x) + \int_{0}^{x} p(x, t) \tilde{k}(x, t, y(t)) dt, \quad x \in [0, T] \]  

(1)

is assumed to satisfy the conditions for a unique solution (see, for example, Tricomi [9] or Smithies [7]). The interval [0, T] is divided into \( N = 2M \) subintervals. The nodes as reported by Attia [3], [4] are chosen to satisfy the following:

\[ 0 = t_0 < t_1 < t_2 \ldots < t_{2M-1} < t_n = T \]

\[ t_{2k} = \left[ \frac{2kT}{N} \right], \quad k = 0, 1, 2, \ldots, M \]

\[ t_{2k+1} = \frac{1}{2} \left[ t_{2k} + t_{2k+1} \right], \quad k = 0, 1, \ldots, M - 1 \]

\[ h_k = t_{k+1} - t_k, \quad k = 0, 1, \ldots, 2M - 1 \]

3. Product Integration: The product-integration method is based on approximating the smooth function \( \tilde{k} \) by a polynomial. More precisely, the integral is approximated by:

\[ \int_{0}^{x} p(x, t) \tilde{k}(x, t, y(t)) dt \approx \sum_{i=1}^{N} w_i \tilde{k}(x_i, t_i, y_i) \]

(2)

where \( x_i = t_i, \quad i = 1, \ldots, N \) and \( p(x, t) = (x - t)^{-n} \) which is the singular part.

The weights depend on the quadrature points \( x_i \). Here, a product integration forms Block a method, approximates the integral term.

3.1. Product Block: In this method equation (1) can be written in the following form:

\[ y(x) = f(x) + \int_{0}^{x} p(x, t) \tilde{k}(x, t, y(t)) dt \]

(3.1)

\[ y(x) = f(x) + \int_{0}^{x} p(x, t) \tilde{k}(x, t, y(t)) dt \]

(3.2)

Then, the above two equations can be written in the following approximated form:

\[ y_{21} = f_{21} + \sum_{i=0}^{31} w_i (x_{21} - t_i)^{-n} \tilde{k}(x_{21}, t_i, y(t_i)) + w_{21,12} \tilde{k}(x_{21}, t_{21}, y_{21}) \]

(4)

and

\[ y_{22} = f_{22} + \sum_{j=0}^{31} \bar{w}_{jj} (x_{22} - t_j)^{-n} \tilde{k}(x_{22}, t_j, y(t_j)) + w_{21,22} \tilde{k}(x_{22}, t_{21}, y_{21}) \]

(5)

The weights can be calculated by the following technique:
3.2. Calculation of weights: In general if \( g(x) \) is singular in \([a, b]\) and \( f(x) \) is a smooth function in the same interval then,

\[
\int_a^b g(x)f(x) \, dx = p_1 f(x_1) + p_2 f(x_2) + \ldots + p_n f(x_n) \quad x_i \in [a, b]
\]

where \( p_1, p_2, \ldots, p_n \) can be calculated exactly from the solution of the system of equations:

\[
\int_a^b g(x)x^{k} \, dx = p_1 x(x_1) + p_2 x(x_2) + \ldots + p_n x(x_n),
\]

\( z(x) = x^k \) and \( k = 0, 1, 2 \).

This method is called Product-3 [2]. Then the weights are:

\[
\int_{x_i}^{x_{i+1}} f(t) \, dt = \sum_{j=1}^{i} w_{x_{j-1}} f_{j-1} + w_{x_{i}} f_{j} + w_{x_{i+1}} f_{i+1}
\]

where:

\[
w_{x_{i+1}} = \frac{aR_{i} - R_{i}(a+b) + bR_{i}}{(c-a)(c-b)}, \quad w_{x_{i-1}} = \frac{a-c}{b-a} w_{x_{i+2}} + \frac{R_{i} - aR_{i}}{b-a} \]

\[
w_{x_{i}} = -w_{x_{i+1}} - w_{x_{i-1}} + R_{i}, \quad \quad \quad R_{i} = \frac{(x_{i} - x_{i-1})^{2}}{1 - \alpha}
\]

\[
R_{j} = (x_{j} - x_{j-1})^{2} \left( \frac{x_{j} - x_{j-1}}{2 - \alpha} - \frac{2x_{i}(x_{j} - x_{j-1})}{2 - \alpha} \right),
\]

where \( a = b_i, b = b_{i-1}, c = b_{i+2} \) and \( v = 2i+1, 2i+2 \) for the first and the second integral respectively. The weights are \( w_{x_{i}}, \bar{w}_{x_{i}} \) can be calculated as follows:

\[
\int_{x_{j-1}}^{x_{j}} f(t) \, dt = \sum_{k=1}^{j-1} w_{x_{k}} f_{k} + \sum_{k=j+1}^{j+1} w_{x_{k}} f_{k} + \sum_{k=j+2}^{j+2} w_{x_{k}} f_{k}
\]

then,

\[
w_{x_{j}} = \frac{abf_{i} - f_{i}(a+b) + f_{i}}{(c-a)(c-b)}, \quad w_{x_{j-1}} = \frac{a-c}{b-a} w_{x_{j+2}} + \frac{f_{i} - aR_{i}}{b-a} \quad \text{and}
\]

\[
w_{x_{j+1}} = -w_{x_{j+2}} - w_{x_{j+1}} + f_{i}
\]

where \( j = 1, 2, 3, \ldots, \ldots, i \), \( a = b_{2j-2}, b = b_{2j-1}, c = b_{2j} \) and

\[
f_{i} = \left( \frac{(x_{i} - x_{j+2})^{2}}{2 - \alpha} - \frac{x_{i}}{2 - \alpha} \right), \quad \bar{f}_{i} = \left( \frac{(x_{i} - x_{j+2})^{2}}{1 - \alpha} - \frac{x_{i}}{1 - \alpha} \right)
\]
\[ I_3 = (x_v - x_z)_{z=1}^{\alpha} + \frac{2x_v(x_v - x_z)}{2 - \alpha} \frac{x_z^2}{3 - \alpha} \]

\[ - (x_v - x_{z=2})_{z=1}^{\alpha} + \frac{2x_v(x_v - x_{z=2})}{2 - \alpha} \frac{x_{z=2}^2}{3 - \alpha} \]

where \( v = 2i + 1 \) for \( w_2 \) and \( v = 2i + 2 \) for \( w_2 \).

Equations (4) and (5) are solved to get \( y_{2n+1}, y_{2n+2} \).

### 3.3 Product Block Algorithm

This algorithm deals with linear Volterra integral equations with singular kernel.

**Algorithm**

To find a solution to linear Volterra integral equations with singular kernel of type \( k(x,t)=(x-t)^{\alpha} \) with variable and constant step length by Product Block.

**INPUT**

Number of intervals (N), Alpha (\( \alpha \)) and Beta (\( \beta \)).

**OUTPUT**

Approximate Solutions \( y_2 \), root mean square error (\( E_{rms} \)), maximum error (\( E_{max} \)) and position of maximum error (\( X_{max} \)).

**Step 1**

\( y[0]=I(0), M=N/2; x[0]=0, t[0]=0 \).

**Step 2**

For \( i=1 \) to \( M-1 \) do steps (3-5).

**Step 3**

\( k=2i, \ x[k]=k/N \).

**Step 4**

\( T[k]=x[k], \quad j=2i-1 \)

\( H[j]=x[j+1]-x[j-1]/2 \)

\( x[[j]]=x[j-1]+H[j] \)

\( T[[j]]=x[j] \).

**Step 5**

\( x[n]=1, \quad T[n]=1, \quad H[M]=(x[n]-x[n-2])/2 \)

\( x[n-1]=x[n-2]+H[M], \quad T[n-1]=x[n-1] \).

**Step 6**

For \( i=0 \) to \( M-1 \) do steps (7-11).

**Step 7**

Calculate \( w[2i+1,2i], w[2i+1,2i+1], w[2i+1,2i+2] \).

\( w[2i+2,2i], w[2i+2,2i+1], w[2i+2,2i+2] \) using Product-3.

**Step 8**

Calculate \( b_1, b_2, b_3, b_4, b_5, b_6 \).

**Step 9**

Define \( L_1 = b_2 \quad b_4 \quad b_5 \quad b_6 \quad b_7 \quad b_8 \quad b_9 \quad b_{10} \)

\( L_2 = b_2 \quad b_4 \quad b_5 \quad b_6 \quad b_7 \quad b_8 \)

\( b = b_2 \quad b_4 \quad b_5 \quad b_6 \quad y[2i+1]-L_1/6 \quad y[2i+1]-L_2/6 \)

**Step 10**

If \( (k=M-1) \) then do steps (11-16).

**Step 11**

\( b_1=0 \).

**Step 12**

For \( l=1 \) to \( 1 \) do steps (13-16).

**Step 13**

Calculate \( w[2i+1,2i], w[2i+1,2i-1], w[2i+1,2i-2] \).

\( w[2i+2,2i], w[2i+2,2i-1], w[2i+2,2i-2] \) using Product-3.

**Step 14**

For \( j=1 \) to \( 1 \) do steps (15-16).

**Step 15**

\( b_3 = b_3 + w[2i+1,2i]\ k(x[2i+1], t[j]) \ y[j] \).

**Step 16**

\( b_3 = b_3 + w[2i+2,2i]\ k(x[2i+2], t[j]) \ y[j] \).

**Step 17**

OUTPUT.

Stop.

For non-linear Volterra integral equations with singular kernel, Newton-Raphson method is used to solve the non-linear equation and step 9 is changed as follow:

**Step 9**

\( p_0=y[2i], \quad q_0=y[2i] \) and call Newton procedure to calculate \( y[2i+1], y[2i+2] \).
4. Numerical Results: The numerical algorithm described above is used for the computation of the Product Block and tested for several examples.

4.1 Test Examples: The following table shows different examples used in Linear case (These examples were taken from [2], [6], [7]).

<table>
<thead>
<tr>
<th>Ex</th>
<th>Equation</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y(x) = x - \frac{4}{3} x^{\frac{1}{3}} + \int_0^1 (x-t)^{\frac{5}{3}} y(t) dt )</td>
<td>( y(x) = x )</td>
</tr>
<tr>
<td>2</td>
<td>( y(x) = f_1(x) - \frac{1}{4} \int_0^1 (x-t)^{\frac{5}{3}} y(t) dt )</td>
<td>( y(x) = \frac{1}{\sqrt{1+x}} )</td>
</tr>
<tr>
<td>3</td>
<td>( y(x) = f_2(x) + \int_0^1 (x-t)^{\alpha} (x^2 + t^2) y(t) dt )</td>
<td>( y(x) = x )</td>
</tr>
<tr>
<td>4</td>
<td>( y(x) = x - \left[ \frac{3}{1-\alpha} - \frac{1}{(2-\alpha)} \right] x^{\frac{1}{2}} + \int_0^1 (x-t)^{\alpha} t y(t) dt )</td>
<td>( y(x) = x )</td>
</tr>
</tbody>
</table>

where
\[
f_1(x) = \frac{1}{\sqrt{1+x}} + \frac{x}{4} - \frac{1}{8} \sin^{-1} \left( \frac{1-x}{1+x} \right) \quad \text{and}
\]
\[
f_2(x) = x - \left[ \frac{1}{(1-\alpha)(2-\alpha)} + \frac{5}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)} \right] x^{\frac{1}{3}}
\]

4.2 Model Integral Equation with Two Types of Singularities:
\[
y(x) = 1 + \frac{\Gamma(3-\alpha)}{2} + \int_0^1 (x-t)^{\alpha-1} y(t) dt
\]
\[
y(x) = 2x^\alpha \quad 0 < \alpha < 1, \quad 0 \leq x \leq 1
\]

4.3 Test Example of Non-Linear Case:
\[
y(x) = x - \frac{6x^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)} + \int_0^1 (x-t)^{\alpha} y^3(t) dt
\]
\[
y_s = x
\]

Results of the root mean square errors, maximum errors and its positions for the different considered examples uses Block method for linear and non-linear cases are shown in Table 2 and Table 3, respectively. \( p(x, t) = (x-t)^{\alpha} \), while the same results using Product Block method for linear case is shown in Table 4 when \( p(x, t) = e^{x-t} (x-t)^{\alpha} \).
Table (2) Results of the root mean square errors, maximum errors and its position, when \( p(x,t) = (x - t)^{\alpha} \) for different examples using Product Block method in Linear case when \( p(x,t) = (x-t)^{\alpha} \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( N )</th>
<th>( \beta E_{\text{rms}} )</th>
<th>( X_{\text{EM}} )</th>
<th>( E_{\text{M}} )</th>
<th>( E_{\text{EM}} )</th>
<th>( E_{\text{rms}} )</th>
<th>( E_{\text{EM}} )</th>
<th>( E_{\text{RMS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0.5</td>
<td>1.35</td>
<td>1.16 \times 10^{-2}</td>
<td>1.000</td>
<td>1.52 \times 10^{-1}</td>
<td>7.82 \times 10^{-1}</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>0.5</td>
<td>1.30</td>
<td>2.47 \times 10^{-8}</td>
<td>0.249</td>
<td>4.62 \times 10^{-4}</td>
<td>1.36 \times 10^{-7}</td>
<td>0.93</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
<td>0.1</td>
<td>1.00</td>
<td>3.35 \times 10^{-7}</td>
<td>0.97</td>
<td>6.38 \times 10^{-5}</td>
<td>6.93 \times 10^{-7}</td>
<td>0.97</td>
</tr>
<tr>
<td>32</td>
<td>0.2</td>
<td>1.00</td>
<td>5.64 \times 10^{-7}</td>
<td>0.97</td>
<td>1.11 \times 10^{-4}</td>
<td>1.4 \times 10^{-7}</td>
<td>0.97</td>
<td>1.09 \times 10^{-4}</td>
</tr>
<tr>
<td>32</td>
<td>0.3</td>
<td>0.95</td>
<td>8.46 \times 10^{-7}</td>
<td>0.97</td>
<td>2.05 \times 10^{-5}</td>
<td>2.05 \times 10^{-7}</td>
<td>0.97</td>
<td>8.46 \times 10^{-7}</td>
</tr>
</tbody>
</table>

where \( E_{\text{rms}} \): Root Mean Square Error, \( E_{\text{M}} \): Maximum Error, \( X_{\text{EM}} \): Position of maximum error and \( E_{\text{R}} = \frac{E_{\text{rms}} \text{ of Graded Mesh}}{E_{\text{rms}} \text{ of Uniform Mesh}} \).

Table (3) Results of the root mean square errors, maximum errors and its position, for different cases of \( \alpha \) using Product Block method in Non-Linear case when \( p(x,t) = (x-t)^{\alpha} \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( N )</th>
<th>( \beta E_{\text{rms}} )</th>
<th>( X_{\text{EM}} )</th>
<th>( E_{\text{M}} )</th>
<th>( E_{\text{EM}} )</th>
<th>( E_{\text{rms}} )</th>
<th>( E_{\text{EM}} )</th>
<th>( E_{\text{RMS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>8</td>
<td>0.95</td>
<td>7.26 \times 10^{-9}</td>
<td>0.380</td>
<td>1.19 \times 10^{-2}</td>
<td>1.39 \times 10^{-1}</td>
<td>0.875</td>
<td>7.69 \times 10^{-4}</td>
</tr>
<tr>
<td>16</td>
<td>0.95</td>
<td>4.70 \times 10^{-4}</td>
<td>0.940</td>
<td>8.72 \times 10^{-5}</td>
<td>1.02 \times 10^{-4}</td>
<td>0.938</td>
<td>4.86 \times 10^{-4}</td>
<td>0.968</td>
</tr>
<tr>
<td>32</td>
<td>0.95</td>
<td>3.96 \times 10^{-7}</td>
<td>0.036</td>
<td>6.25 \times 10^{-5}</td>
<td>7.14 \times 10^{-7}</td>
<td>0.969</td>
<td>3.13 \times 10^{-5}</td>
<td>0.978</td>
</tr>
<tr>
<td>64</td>
<td>0.95</td>
<td>1.99 \times 10^{-9}</td>
<td>0.018</td>
<td>4.79 \times 10^{-9}</td>
<td>4.80 \times 10^{-9}</td>
<td>0.984</td>
<td>2.02 \times 10^{-9}</td>
<td>0.984</td>
</tr>
<tr>
<td>0.2</td>
<td>8</td>
<td>0.95</td>
<td>1.06 \times 10^{-6}</td>
<td>0.880</td>
<td>1.77 \times 10^{-4}</td>
<td>2.08 \times 10^{-4}</td>
<td>0.875</td>
<td>1.14 \times 10^{-4}</td>
</tr>
<tr>
<td>16</td>
<td>0.95</td>
<td>7.92 \times 10^{-5}</td>
<td>0.940</td>
<td>1.40 \times 10^{-4}</td>
<td>1.67 \times 10^{-4}</td>
<td>0.938</td>
<td>7.44 \times 10^{-4}</td>
<td>0.944</td>
</tr>
<tr>
<td>32</td>
<td>0.95</td>
<td>4.76 \times 10^{-7}</td>
<td>0.970</td>
<td>1.03 \times 10^{-4}</td>
<td>1.24 \times 10^{-4}</td>
<td>0.969</td>
<td>4.96 \times 10^{-4}</td>
<td>0.960</td>
</tr>
<tr>
<td>64</td>
<td>0.95</td>
<td>3.26 \times 10^{-8}</td>
<td>0.017</td>
<td>7.74 \times 10^{-8}</td>
<td>8.78 \times 10^{-8}</td>
<td>0.984</td>
<td>3.36 \times 10^{-8}</td>
<td>0.972</td>
</tr>
<tr>
<td>0.3</td>
<td>8</td>
<td>0.95</td>
<td>1.72 \times 10^{-5}</td>
<td>1.000</td>
<td>3.09 \times 10^{-4}</td>
<td>4.85 \times 10^{-4}</td>
<td>1.000</td>
<td>2.22 \times 10^{-4}</td>
</tr>
<tr>
<td>16</td>
<td>0.95</td>
<td>1.19 \times 10^{-4}</td>
<td>0.077</td>
<td>2.41 \times 10^{-5}</td>
<td>3.33 \times 10^{-5}</td>
<td>0.938</td>
<td>1.43 \times 10^{-5}</td>
<td>0.835</td>
</tr>
<tr>
<td>32</td>
<td>0.95</td>
<td>8.84 \times 10^{-7}</td>
<td>0.970</td>
<td>2.28 \times 10^{-5}</td>
<td>2.77 \times 10^{-5}</td>
<td>0.969</td>
<td>9.61 \times 10^{-5}</td>
<td>0.919</td>
</tr>
<tr>
<td>64</td>
<td>0.95</td>
<td>6.31 \times 10^{-8}</td>
<td>0.985</td>
<td>1.76 \times 10^{-7}</td>
<td>2.10 \times 10^{-7}</td>
<td>0.984</td>
<td>6.73 \times 10^{-8}</td>
<td>0.938</td>
</tr>
</tbody>
</table>
In the following figure the relation between $-\log(E_{rms})$ and $\beta$ is plotted at $N = 32$, when $p(x,t) = t^{\alpha-1}(x-t)^{\alpha}$ and $\lambda = 0.2, -5, 1$.

*The maximum values of $-\log(E_{rms})$ means minimum value of $E_{rms}$
*When $\lambda = 0.2$ the minimum $E_{rms}$ occurs at $\beta = 0.90$
*When $\lambda = -5$ the minimum $E_{rms}$ occurs at $\beta = 0.80$
*When $\lambda = 1$ the minimum $E_{rms}$ occurs at $\beta = 1.15$

5. Conclusions: The product methods give good results when using it on graded mesh compared with equal spaced nodes at the same number of subintervals. We use the product Block to solve singular Volterra integral equation of the second kind. We conclude to use product Block on graded mesh. The equal spaced nodes can be considered as a special case from the graded mesh. Alia and Nour (3) proposed that the value of $\beta$ is controlled by two factors: the first factor is the suprimum of the function and the second is the large variation of the function. For the second case, the kernel of this type has two types of singularities, the first at $t = 0$ and the other is moving along the line $x = t$. Results of experiment showed that:
1. The product method is a good method to solve this type of singular equations.
2. The optimum value of $\beta$ in the case of graded meshes is $\beta \in (0.7, 1.5)$.
6. References:
(9) Tricomi, F.G., (1957), "Integral Equations", Inter Science, New York