ELASTIC STABILITY OF STEEL STRUCTURAL SYSTEMS RESTRAINED BY INITIALLY CURVED MEMBERS

BY
DR. AHMED BADR

ABSTRACT
The axial stiffness of elastic, initially curved, members is considered using small-deformation theory. Simple relations are obtained for the tangent axial stiffness. The use of this equivalent spring stiffness is described in an attempt to provide a unified description of the restraint given to a main column member by the provision of initially curved members at mid-height. The buckling of a column element in the plane of the nonlinear restraints is investigated. Further the purpose of the analysis is to describe precise values to the spring stiffness in the light of the analysis referred to the above. Further, the matrix technique illustrated in the present paper is capable of generalization, especially using automatic computational procedures. Finally the critical load of the structural system is investigated with respect to the presence of imperfection in the lateral restraints.

INTRODUCTION
The nonlinear analysis of structural systems consisting of main individual columns stiffened laterally by initially curved members are considered. Some unavoidable imperfection are taken into account and a critical state, still in the elastic range, is examined. The basic equations of the axial stiffness of elastic, initially curved, members are reviewed especially to provide information on the equivalent spring stiffness of this type of members. Simple relations are obtained for the tangent axial stiffness in terms of the slenderness ratio, initial curvature and axial thrust in the member. Then the use of this equivalent spring stiffness is described in an attempt to provide a unified description of the restraint given to a main column member by the provision of initially curved members at mid-height. The reduction of the critical load is investigated with respect to the presence of imperfection in the lateral restraints.

1: THE AXIAL STIFFNESS OF IMPERFECT CURVED, COMPRESSION MEMBERS.
Considering a compression member that has initial curvature of sine wave function as shown in Fig.(1) and given by:
where \( y_o \) is the initial deflection of the member from its chord-line position, \( a_o \) is the amplitude and \( l \) is the chord line length at the initial curved position.

Further, the effect of compressive axial load upon this initial shape is to amplify the initial deflections such that the final deflected shape \( y \) is easily obtained [6].

\[
y = \frac{a_o}{l} \sin \left( \frac{n_x}{l} \right) \quad 0 < x < l \quad \text{.........(1)}
\]

\[
y = \frac{a_o}{l} \sin \frac{n_x}{l} \quad \text{.........(2)}
\]

in which: \( P_E = \frac{a_o}{l} \) is the Euler buckling load when the member is perfectly straight, and \( l \) is the chord line length of the initially curved member.

Now Eq.(2) may be used to evaluate the shortening \( \Delta \) of the chord-line of the curved member due to the compression applied load \( P \)

\[
\Delta = \Delta_b + \Delta_a \quad \text{.........(3)}
\]

where, \( \Delta_b \) is the chord-line shortening due to bowing and \( \Delta_a \) is the axial shortening of the member due to direct stress (i.e. due to the axial force \( P \)) in which:

\[
\Delta_b = \frac{1}{2} \int_a^b \frac{3a_o^2}{l^2} \sin^2 \left( \frac{n_x}{l} \right) dx - \frac{1}{2} \int_a^b \frac{a_o^2}{l^2} \sin \left( \frac{n_x}{l} \right) dx
\]

using Eqs.(1) and (2) for the values of \( y_o \) and \( y \), then:

\[
\Delta_b = \frac{a_o}{l^2} \frac{n_x}{l} \left( \frac{n_x}{l} \right) \int_0^l \cos \left( \frac{n_x}{l} \right) dx \quad \text{where} \mu = P/P_E
\]

\[
\Delta_b = \frac{a_o}{l^2} \frac{n_x}{l} \left( \frac{n_x}{l} \right) \int_0^l \cos \left( \frac{n_x}{l} \right) dx
\]

or \( \Delta_b = \frac{a_o}{l^2} \left( \mu^2 - \mu \right) / 4 \mu \mu \text{.........(4)} \)

and \( \Delta_a = \frac{P}{AE} \left( l + \frac{\mu^2}{4} \right) \text{.........(5)} \)

Since the actual length of the unloaded member is approximately equal to \( (l + a_o/4l) \), therefore

\[
\Delta_l = \frac{a_o}{l^2} \frac{n_x}{l} \left( \mu^2 - \mu \right) + \frac{P}{AE} \left( l + \frac{\mu^2}{4l} \right) \text{.........(6)}
\]

It is convenient to express Eq.(6) in non-dimensional form as follows:
let \( \Delta_t = \Delta / \lambda \) and \( \frac{P}{AE} = \mu \frac{n^2}{\lambda^2} \)

where, \( \lambda \) is the slenderness ratio \((1/r)\) of the element, and \( r \) is the radius of gyration.

Then Eq. (6) may be re-written as follows:

\[
\Delta_t = \frac{\pi^2 \mu (2-\mu)}{1+\mu} \left( 1 - \frac{3n^2}{4} \right) \frac{\Delta}{\lambda^2}
\]

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\]

The axial stiffness of a curved member \( K^* \) can be defined as the load applied along the chord-line of the member, which will produce a total axial shortening of the chord-line equal to unity, i.e.

\[ P = K^* \Delta_t \]

where: \( P \) is the axial load on the column and \( \Delta_t \) is the total shortening or total elongation of the chord-line.

In general \( K^* \) will be a function of \( P \), but will reduce to the "Hooke's law" stiffness \( EA/\lambda \) when the effect of bowing is ignored. Therefore:

\[ K^* = \frac{P}{\Delta_t} \text{ or in non-dimensional form:} \]

\[
K^* = \frac{\mu}{\frac{\pi^2}{1+\mu} \left( 1 - \frac{3n^2}{4} \right) \frac{\Delta}{\lambda^2}} \]

Eq. (6) gives the non-dimensional tangent axial stiffness of the curved member at any load level \( \mu \) in terms of the initial amplitude of the curved shape, \( \varepsilon \), and the geometrical properties of the member. The initial stiffness of the unloaded curved member \( \mu=0 \) follows directly from Eq. (6) by differentiation of the numerator and denominator with respect to \( \mu \), and substitution of \( \mu=0 \) then:

\[
K^* = \frac{\pi^2}{1+\mu} \left( 1 - \frac{3n^2}{4} \right) \frac{\Delta}{\lambda^2}
\]

if the initial curvature \( \varepsilon \) is zero, then \( K^* = \frac{\pi^2}{1+\mu} \left( 1 - \frac{3n^2}{4} \right) \frac{\Delta}{\lambda^2} \)

i.e. \( K^* \) reduces to \( EA/\lambda \) "the Hooke's law" stiffness.

Eq. (9) yields the equivalent axial spring stiffness of an initially curved member when used as a lateral restraint against buckling of a main column leg. By inspection in Eq. (9) it is seen that, the limiting value of \( K^*/P \) i.e. the tangent axial stiffness in non-dimensional form at \( \mu=0 \), reduces while the amplitude of the curved shape, \( \varepsilon \), increases at limiting value of \( \lambda \).

The effect of axial load \( P \) on the non-dimensional tangent axial stiffness \( K^*/P \), for various values of \( \lambda \), when \( \varepsilon = 0.003 \) by way of example, is shown in Fig. (2).

The chart was obtained by plotting, on horizontal axis, values of the non-dimensional tangent axial stiffness \( K^*/P \), and, on vertical axis, values of the slenderness ratio \( \lambda \). It is seen that, from the diagrams shown in Fig. (2), the axial stiffness \( K^* \) reduces while the \( \mu \) value (i.e. the applied load \( P \)) increases at limiting value of curvature \( \varepsilon \).

2: THE ELASTIC BUCKLING LOADS OF A STRUCTURAL SYSTEM RESTRAINED LATERALLY BY INITIALLY CURVED MEMBER
Consider a simple problem of a main column leg pinned at its ends, and restrained laterally by an elastic curved member pinned to the column at mid-height and pinned to a rigid support at the other end, as shown in Fig. (3). The problem is easily dealt with a matrix technique similar to that employed in a general buckling analysis.

Now let \( \{ R \} \) be the set of exciting forces, applied to the node 1, Fig. (3), and \( \{ r \} \) the corresponding set of node displacements. Then, the governing equation for the behaviour of the system may be written as:

\[
\{ R \} = [K] \{ r \} \]

where: \([K]\) is the overall stiffness matrix of the spring-supported column, and in this case is an assembly of the member stiffness matrices for the elements "a" and "b" together with the contribution from the equivalent spring stiffness of the curved member.

Eq. (10) is an entirely general result in which the vector \( \{ R \} \) is a column matrix of all relevant node forces and \( \{ r \} \) is the corresponding vector of node displacements. The presence of spring supports can always be incorporated by addition of appropriate spring stiffness to the corresponding elements of the leading diagonal of \([K]\). Ignoring this contribution from the spring constraint, and referring to the elements "a" and "b" using the conventional notation of end 1 and end 2, as defined in Fig. (2) it follows that, in the simple structure under consideration

\[
[K] = [k_{aa}] + [k_{bb}] \]

where: \([k_{aa}] \) and \([k_{bb}] \) are the elements stiffness sub-matrices abstracted from a general statement of an element force-displacement relationship of the form:

\[
\{ F \} = \begin{bmatrix} k_{aa} & k_{ab} \\ k_{ba} & k_{bb} \end{bmatrix} \{ u \} \]

where: \(F, u\) are member forces and member displacements respectively.

It should be noted that, in this case, \([k_{aa}]\) and \([k_{bb}]\) are member stiffness sub-matrices appropriate to a beam-column element pinned to a rigid support at end 1 and 2 respectively, and allow for the presence of axial load \( P \) in each element.

The overall stiffness matrix \([K]\) for this particular structure can be obtained and derived as follows:

\[
\begin{bmatrix}
4EA & 0 & 0 & 0 \\
0 & 4EI & 0 & 0 \\
0 & 0 & 4EI & 0 \\
0 & 0 & 0 & \frac{El}{c} \\
\end{bmatrix}
\]

where: \(K^*\) is the equivalent spring stiffness of the curved member \( E I \) is the flexural rigidity of the main column.
Ac. \( a \) are the cross sectional area and the overall length of the column respectively, and \( \alpha_1, \alpha_2 \) are stability functions dependent upon the axial load in the column and are defined as follows:

\[
\begin{align*}
\alpha_1 &= \sqrt{1 - c^2} - \frac{2}{c} \mu_m \\
\alpha_2 &= \sqrt{1 - c^2}
\end{align*}
\]  

(14)

where: \( \mu_m = \frac{P}{P_e} = \frac{P}{c_0^2/\pi^2 \xi_1} \) is the ratio of the axial load in a column element to the Euler load of the column element.

and \( S = \frac{\eta (1 - 2 \beta \cos 2\beta)}{\tan \beta - \beta} \)

\[
C = \frac{2 \beta - \eta \cos \beta}{\sin 2\beta - 2 \beta \cos 2\beta}
\]

which \( \beta = \pi/2 \sqrt{\mu_m} \)

The general condition for structural instability is that \( |K| = 0 \) from which the following buckling criteria results:

\[
\begin{align*}
\alpha_1 &= 0 \\
\alpha_2 &= \frac{\kappa^*}{1 + \kappa^*}
\end{align*}
\]  

(15)  

(16)

This result is identical in form to that derived by Timoshenko and Gere [12]. The only merit of the analysis shown above is in that the technique is entirely general for any elastic spring-supported assemblage of elements and may be automated for a computer solution.

The investigation is typical of that which could be carried out when the column is constrained laterally by a number of curved members pinned to it at various points along its length.

Equations (15) and (16) define completely the occurrence of all in-plane buckling loads. There will be an infinite number of such loads and generally it is the first buckling load which is of practical significance. Examination of these equations gives the full account of the buckling behaviour, interpreted in terms of the geometrical properties of the side curved member and main column. Firstly examining Eq. (15) and writing \( \alpha_2 \) in terms of \( \beta \) using the results quoted above for \( S \) and \( C \); the equation may be re-written in the form:

\[
\begin{align*}
\alpha_2 &= \frac{2 \beta - \eta \cos \beta - 2 \beta \cos 2\beta}{1 - \beta \cot \beta} \left( \frac{1 - \beta \cot \beta}{(2 \beta - 2 \beta \cos 2\beta)} \right) = 0
\end{align*}
\]  

(17)

Inspection of Eq. (17) it is seen that:

\[ \alpha_2 = 0 \]  

when \( \beta = \pi/2 \) i.e.  

\[ \alpha_2 = 0 \]  

when \( \mu_m = 1 \) and the initially buckled shape is the anti-symmetric one shown in Fig. (4-a). This may \( \alpha_2 \) may not be the first critical buckling load dependent upon the value of \( K \) resulting from Eq. (16).

Now proceed to an examination of Eq. (16), and substitute for \( \kappa^* \) from
Eq. (9), after re-arrangement of the algebra, making use of the non-dimensional quantities \( \lambda_c \) and \( \lambda_e \) the buckling criteria of Eq. (16) may be written in the following form.

The buckling occurs when:

\[
\alpha_1 = \frac{\lambda_e^2}{4} \frac{\lambda_c}{\lambda_e^2 + \epsilon \lambda_c^2 + \epsilon \lambda_e^2 + 4}
\]

(18)

and this criterion is conveniently written, entirely non-dimensionally, as:

\[
\alpha_1 = \frac{\lambda_e^2}{2 \lambda_c^2 + \epsilon \lambda_c^2 + \epsilon \lambda_e^2 + 4}
\]

(19)

where, \( \nu = \frac{\lambda_c^2}{4 \lambda_e^2} \)

(20)

\( \lambda_e \) is the slenderness ratio of the side curved member

\( \lambda_c \) is the slenderness ratio of the column

The slenderness ratios are defined as the actual length of the element divided by the appropriate radius of gyration.

It should be noted that \( \alpha_n \) is a function of the axial load on the main column and Eq. (19) may yield either the first or the second buckling mode. The mode shape is shown in Fig. (4-b) where the lateral displacement of node i depends upon the spring stiffness \( K \).

The results of the above analysis can be conveniently represented graphically in chart form giving an automatic means of determining the critical buckling loads from Eq. (19), at a glance.

The behaviour is summarised in Fig. (5) for a range of laterally curved member slenderness ratio \( \lambda_e \) from 100 to 350 and a selection of values of the non-dimensional parameter \( \nu \). Two values of the initial curvature, \( \epsilon \), of the lateral restraint are shown viz. \( \epsilon = 0.003 \) and \( \epsilon = 0.01 \). Similar charts could, of course, be plotted for other initially curved shapes. The diagram was obtained by plotting, on a horizontal axis, values of the stability function \( \alpha_c \), for various values of \( \mu_e \) and, on a vertical axis, the ratio of the axial load in the column to the Euler load of the complete column \( (\mu_e = 4\mu) \), and the right hand side of Eq. (19) for various values of \( \nu \), \( \epsilon \) and \( \lambda_e \).

To use the chart in Fig. (5), locate the slenderness ratio value of the lateral restraint on the right hand side vertical axis. Knowing the value of curvature \( \epsilon \), then \( \nu \) value can be calculated from Eq (20). The chart is traversed horizontally to meet the appropriate \( \nu \) curve, thence vertically to intercept the single \( \alpha_c \) plot, and therefore the horizontal traverse is completed to intercept the left hand side vertical axis at the critical value of \( \mu_e \).

It is seen that the chart in Fig. (5) is constructed specifically in terms of geometrical properties and imperfections of the side curved member. Also this chart predominantly yields values of \( \mu_e < 4.0 \) and indicates that Eq. (19) picks out the first critical buckling mode.

Finally, the relative importance of Eq. (15) and (16) in picking out the
first critical buckling mode may be summarised in one simple interaction
diagram between $\nu$ and $\epsilon$, for any value of $\lambda*$. By substituting the value of $\alpha$ at $\mu_0 = 1.0$ in Eq. (19) and plotting the resulting equation in $\nu$ and $\epsilon$.

Since $\alpha = \pi(1 - \nu^2) - n^2 \mu_0$ and it may be shown that $\pi(1 - c^2) = 0$ when $\mu_0 = 1.0$, then this condition. Eq.(19) reduces to:

$$n^2 = \nu / (2\nu^2 + \pi^2 \nu^2 + 4)$$

$$\nu = n^2 (2\nu^2 + \pi^2 + 4)$$

..............................(21)

This result is plotted in Fig.(5), for a range of slenderness ratio of the side rail between 100 and 500. To determine whether the first critical buckling load of the column lies below or above $\mu_0 = 4$, $(\mu_0 = 1.0)$, the interaction diagram is entered with pre-determined values of $\nu$ and $\epsilon$ (which are functions of the geometry of the system only). If the entry point lies to the left of the curve corresponding to the appropriate value of $\lambda*$, then the first critical buckling load $\mu_0$ is less than 4.0 and will be given by Fig.(5). Other wise the first critical buckling load will be $\mu_0 = 4.0$. The second critical buckling load, which is, of course, largely of theoretical interest only, can never exceed $\mu_0 = 0.184$ since this corresponds to the mode shown in Fig.(4-c) and requires an infinitely stiff constraint at mid-height of the column.

CONCLUSIONS

An attempt has been made to assess the axial spring stiffness of curved elements end to apply the results to one fairly simple constraint problem. Whilst the problem chosen has fair resemblance to a particular practical condition, the main purpose in presenting the above work has been to provide a unified view of the elastic buckling behaviour of the system considered. The presentation of information in chart form has the merit that the results of the analysis can be ascertained with a minimum of involvement with the algebra, and does completely describe the fundamental behaviour concisely in a situation where there are many design parameters.

There seems to be no overwhelming difficulty in presenting similar charts for more complex situations containing a number of spring constraints and even including the effects of lateral-torsional buckling of elasto-plastic behaviour.

REFERENCES


(a) Loaded strut  

(b) Initially unloaded curved member  

FIG.(1)

![Graph showing the effect of axial load on the non-dimensional tangent axial stiffness.](image)

**FIG.(2).** THE EFFECT OF AXIAL LOAD "P" ON THE NON-DIMENSIONAL TANGENT AXIAL STIFFNESS
(a) Schematic Description of Problem

(b) Mathematical Model

\[ \mathbf{R} = [R_x, R_y, M] \]
\[ \mathbf{r} = [r_x, r_y, \theta] \]

FIG.(3)

(a) \( \mu_c = 4.0 \)  
(b) \( \mu_o = \frac{K^{c,c}}{4EI} \)  
(c) \( \mu_o = 0.104 \)

FIG.(4) CRITICAL BUCKLING MODES
Fig. (5). Typical Chart giving First Critical Buckling Loads of Main Columns
\[ \psi = \lambda \frac{\lambda_x^2}{\rho^2} + 4 \lambda \frac{\lambda_z}{\rho^2} \]

Fig.(6)